FAQs & their solutions for Module 1:
Introduction & Basic Mathematical Preliminaries

Question1: An electron of energy 200 eV is passed through a circular hole of radius 10^{-4} cm. What is the uncertainty introduced in the angle of emergence?

Solution1: $$p \approx \sqrt{2mE} = [2\times 0.9\times 10^{-27}\times 3.2\times 10^{-10}]^{1/2} \simeq 8\times 10^{-19} \text{ g cm/sec}$$

Now

$$\Delta p \sim \frac{\hbar}{\Delta x} \simeq \frac{10^{-27} \text{ erg sec}}{2\times 10^{-4} \text{ cm}} = 5\times 10^{-24} \text{ g cm/sec.}$$

$$\theta \sim \frac{\Delta p_x}{p} \approx 6\times 10^{-6} \text{ radians } \simeq 1 \text{ sec of arc}$$

Question2: In continuation of the previous problem, what would be the corresponding uncertainty for a 0.1 g lead ball thrown with a velocity 10^3 cm/sec through a hole 1 cm in radius?

Solution2:

$$p \approx 10^2 \text{ cm g/sec,} \quad \Delta p \approx \frac{\hbar}{\Delta x} \approx 5\times 10^{-28} \text{ g cm/sec.}$$

$$\theta \sim 5\times 10^{-30} \text{ radians } \approx 10^{-34} \text{ sec of arc}$$

Question3: Prove the following representation of the Dirac delta function:

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm ik(x-a)}\, dk \qquad (1)$$

Solution3:

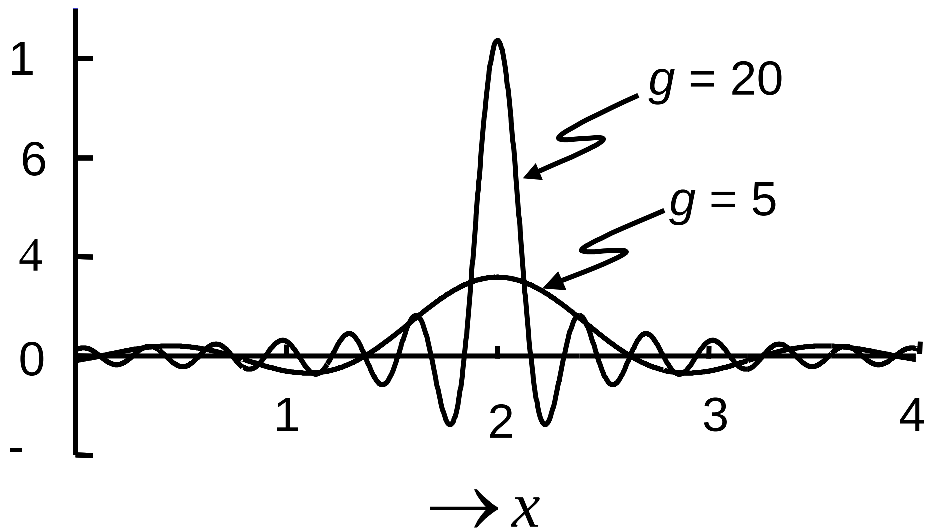
Since $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin\, gx}{\pi\, x}\, dx = 1$ for all values of $g > 0$.

Thus, $$\int_{-\infty}^{+\infty} \frac{\sin\left[g\ \ x-a\ \right]}{\pi\ \ x-a}\, dx = 1$$

The function $\frac{\sin[g(x-a)]}{\pi(x-a)}$ is plotted below for $g = 5$ and $g = 20$. As the value of g increases, it becomes more and more sharply peaked at x=a=2.. Thus the function $\frac{\sin[g(x-a)]}{\pi(x-a)}$ has all the properties of the delta function and hence

$$\delta(x-a) = \lim_{g\to\infty} \frac{\sin[g(x-a)]}{\pi(x-a)}$$

$$\frac{\sin[g(x-2)]}{\pi(x-2)}$$



Now $$\frac{\sin[g(x-a)]}{\pi(x-a)} = \frac{1}{2\pi}\int_{-g}^{+g} e^{\pm ik(x-a)}dk .$$

Thus $$\delta(x-a) = \lim_{g\to\infty}\frac{\sin[g(x-a)]}{\pi(x-a)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm ik(x-a)}dk$$

<u>Question4:</u> Using Eq.(1), show that if we define

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\, e^{\pm ikx}\, dx \qquad (2)$$

then $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k)\, e^{\mp ikx}\, dk \tag{3}$$

The function $F(k)$ is the Fourier transform of the function $f(x)$.

Solution4: Since $f(x) = \int_{-\infty}^{+\infty} \delta(x-x') f(x') dx'$ we may write

$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{\pm ik(x-x')} f(x') dx' dk$ which is known as the Fourier Integral theorem.

Thus if we define

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx}\, dx$$

Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k) e^{+ikx}\, dk$$

The function F(k) is the fourier transform of the function f(x).

Question5: We define $$G_\sigma(x) \equiv \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right);\ \sigma > 0 \tag{4}$$

Show that $\delta(x-a) = \lim_{\sigma\to 0} G_\sigma(x)$ which is the Gaussian representation of the Dirac-delta functions.

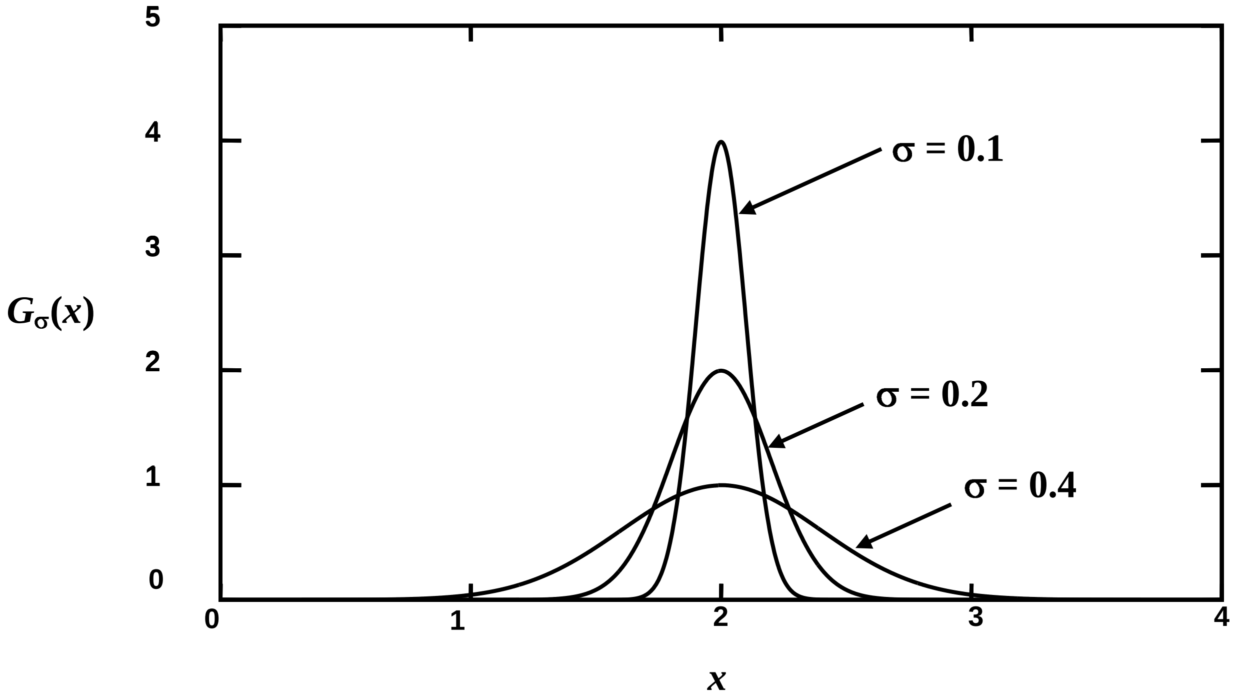
Solution5: We have the integral:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \quad \Rightarrow \quad \int_{-\infty}^{+\infty} e^{-\alpha x^2 + \beta x}\, dx = \sqrt{\frac{\pi}{\alpha}}\ \exp\left[\frac{\beta^2}{4\alpha}\right];\ \operatorname{Re}(\alpha) > 0$$

Now $G_\sigma(x) \equiv \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-a)^2}{2\sigma^2}\right]$; $\sigma > 0$ If we use the above integral we readily obtain:

$\int_{-\infty}^{+\infty} G_\sigma(x)\, dx = 1$. If we plot $G_\sigma(x)$ as a function of x for differenet values of σ (see diagram below) we will find that in the limit of $\sigma \to 0$, the function $G_\sigma(x)$ has all the properties of the Dirac delta function and we have $\delta(x-a) = \lim_{\sigma\to 0} G_\sigma(x)$.

Gaussian Representation of the Delta Function



Question6: Consider a Gaussian pulse given by $f(t) = A\, e^{-t^2/2\tau^2} e^{-i\omega_0 t}$. Calculate its frequency spectrum and show that $\tau\, \Delta\omega \approx 1$.

Solution6: $f(t) = Ae^{-t^2/2\tau^2} e^{-i\omega_0 t}$ Now $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{i\omega t}\, dt$. Thus

$$F(\omega) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2\tau^2}\, e^{i(\omega-\omega_0)t}\, dt$$

If we use the equation $$\int_{-\infty}^{+\infty} e^{-\alpha x^2 + \beta x}\, dx = \sqrt{\frac{\pi}{\alpha}}\, \exp\left[\frac{\beta^2}{4\alpha}\right]; \quad \mathrm{Re}(\alpha) > 0$$

we would readily get

$$F(\omega) = A\sigma \exp\left[-\frac{(\omega-\omega_0)^2 \tau^2}{2}\right].$$

The duration of the pulse is $\sim \tau$ and the frequency spread $\Delta\omega$ is $\sim \frac{1}{\tau}$.

Question7: Show that $\delta(x-a) = H'(x-a)$ where $H(x-a)$ is the unit step function at $x=a$.

Solution7: $\delta(x-a) = \lim_{\sigma\to 0} R_\sigma(x)$; where

$$R_\sigma(x) = \frac{1}{2\sigma} \quad \text{for} \quad -\sigma < x-a < \sigma$$

$$= 0 \quad \text{for} \quad |x-a| > \sigma$$

Consider the ramp function

$$F_\sigma(x) = \frac{1}{2\sigma}(x-a+\sigma) \quad \text{for} \quad -\sigma < x-a < \sigma$$

$$= 0 \quad \text{for} \quad |x-a| > \sigma$$

It can easily be seen that $R_\sigma(x) = \frac{d F_\sigma(x)}{dx}$. In the limit of $\sigma \to 0$, the function $F_\sigma(x)$ becomes the unit step function (see Figure below) --- hence $\delta(x-a) = H'(x-a)$.